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SOME NON-ALIASING RELATIONSHIP FOR SECOND-ORDER MODEL

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ABSTRACT : We consider the second-order model based on a design which is derived from a balanced array of strength 4 and 3 symbols. In this model, when the information matrix of a design is singular, we present some non-aliasing relationship among the factorial effects not to be negligible.

1. Introduction

In a practical experimentation, the most interesting factorial effects are the main effects, the next are the two-factor interactions, and so on. Thus the experimenter want to carry out the experimentation such that the main effects are not confounded (or aliased) with each other, and furthermore that if they are confounded with some effects, then these are possibly higher order interactions which may be negligible. In a (fractional) factorial experiment, the aliasing (or confounding) relationship among the factorial (and/or block) effects has been studied as the defining relationship (e.g., Finney [2]). The extended concept of resolution for 2^m factorials (e.g., Yamamoto and/or Hyodo [5,13]) and balanced fractional 2^m factorial (2^m-BFF) designs of even resolution (e.g., Shirakura [9,10]) can be regarded as the aliasing relationship in a certain sense.

The characteristic polynomial of the information matrix for the second-order model and the economical second-order designs of 3^m factorials were presented by Hoke [3,4]. The second-order model based on 3^m factorials contains the general mean, the linear and the quadratic components of the main effects and the linear by linear ones of the two-factor interactions. Under some conditions, a balanced array (B-array) yields a balanced design (e.g., Kuwada [7]). By using the algebraic structure of the multidimensional relationship, Kuwada [8] obtained an explicit expression for the characteristic polynomial of the information matrix of 3^m-BFF designs of resolution V derived from B-arrays of strength 4. The inversion of the information matrix of 3^m-BFF designs of resolution V was presented by Srivastava and Ariyaratna [11] using the another technique. Optimal 3^m-BFF designs of resolution V were independently obtained by Ariyaratna [1] and Kuwada [6]. An expression for the trace of the variance-covariance matrix of a balanced (2,0)-symmetric design of resolution V for 3^m factorials was also obtained by Srivastava and Chopra [12].

In this paper, attention is focused on finding some non-aliasing relationship for the second-order model when the information matrix of a 3^m-BFF design derived from a B-array of strength 4 is singular. In this situation, there are three cases: (A) the general mean and all main effects are estimable and are not confounded with the two-factor interactions, (B) all main effects are estimable and are not confounded with the general mean and the two-factor interactions, (C) the linear components of the main effects are estimable and are not confounded with the general mean, the quadratic ones of the main effects and the two-factor interactions.

2. Preliminaries

Let $\boldsymbol{\Theta}_0$ and $\boldsymbol{\Theta}_1$ be an $n_0 \times 1$ vector of the factorial effects to be estimated and an $n_1 \times 1$ one not of interest and not assumed to be known, respectively, in the absence of the remaining factorial effects. Further let $\boldsymbol{y}(T)$ be a vector of N observations based on a fraction T with $m(\geq 4)$ factors. Then the linear model may be written as

$$\mathcal{E}[\boldsymbol{y}(T)] = E_0\boldsymbol{\Theta}_0 + E_1\boldsymbol{\Theta}_1 \quad \text{and} \quad \mathrm{Var}[\boldsymbol{y}(T)] = \sigma^2 I_N, \tag{2.1}$$

where E_i (i=0,1) are $N \times n_i$ submatrices of the design matrix $[E_0; E_1](= E_T$, say). Here $\mathcal{E}[\boldsymbol{y}]$ and $\mathrm{Var}[\boldsymbol{y}]$ denote the expected value and the variance-covariance matrix of a random vector $\boldsymbol{y}$, respectively, and I_p is the identity matrix of order p. The normal equation for estimating $(\boldsymbol{\Theta}_0'; \boldsymbol{\Theta}_1')$ $(= \boldsymbol{\Theta}'$, say) is given by

$$M_{00}\hat{\boldsymbol{\Theta}}_0 + M_{01}\hat{\boldsymbol{\Theta}}_1 = E_0'\boldsymbol{y}(T) \quad \text{and} \quad M_{10}\hat{\boldsymbol{\Theta}}_0 + M_{11}\hat{\boldsymbol{\Theta}}_1 = E_1'\boldsymbol{y}(T), \tag{2.2}$$

where $M_{ij}=E_i'E_j$ (i,j=0,1). Throughout this paper, we assume that M_{00} is nonsingular because we want at least to estimate $\boldsymbol{\Theta}_0$. Then it follows from (2.2) that

$$\hat{\boldsymbol{\Theta}}_0 = M_{00}^{-1}E_0'\boldsymbol{y}(T) - M_{00}^{-1}M_{01}\hat{\boldsymbol{\Theta}}_1$$

and

$$\hat{\boldsymbol{\Theta}}_1 = \begin{cases} (M_{11}-M_{10}M_{00}^{-1}M_{01})^{-1}(E_1'-M_{10}M_{00}^{-1}E_0')\boldsymbol{y}(T) & \text{if } \det(M_{11}-M_{10}M_{00}^{-1}M_{01}) \neq 0, \\ (M_{11}-M_{10}M_{00}^{-1}M_{01})^{g}(E_1'-M_{10}M_{00}^{-1}E_0')\boldsymbol{y}(T) & \\ \quad + \{I_{n_1}-(M_{11}-M_{10}M_{00}^{-1}M_{01})^{g}(M_{11}-M_{10}M_{00}^{-1}M_{01})\}\boldsymbol{z} & \text{if } \det(M_{11}-M_{10}M_{00}^{-1}M_{01}) = 0, \end{cases}$$

where $\det(A)$ and A^g denote the determinant of a matrix A and a generalized inverse of a matrix A, i.e., $AA^gA=A$, respectively, and $\boldsymbol{z}$ is an arbitrary vector of size $n_1\times 1$. If $\det(M_{11}-M_{10}M_{00}^{-1}M_{01})\neq 0$, then $\boldsymbol{\Theta}_0$ and $\boldsymbol{\Theta}_1$ can be estimated separately . Thus in this paper, we consider the situation in which $\det(M_{11}-M_{10}M_{00}^{-1}M_{01})=0$. Then we get

$$\hat{\boldsymbol{\Theta}}_0 = M_{00}^{-1}E_0'\boldsymbol{y}(T) - M_{00}^{-1}M_{01}(M_{11}-M_{10}M_{00}^{-1}M_{01})^{g}(E_1'-M_{10}M_{00}^{-1}E_0')\boldsymbol{y}(T)$$
$$- M_{00}^{-1}M_{01}\{I_{n_1}-(M_{11}-M_{10}M_{00}^{-1}M_{01})^{g}(M_{11}-M_{10}M_{00}^{-1}M_{01})\}\boldsymbol{z}, \tag{2.3a}$$

$$\hat{\boldsymbol{\Theta}}_1 = (M_{11}-M_{10}M_{00}^{-1}M_{01})^{g}(E_1'-M_{10}M_{00}^{-1}E_0')\boldsymbol{y}(T)$$
$$+ \{I_{n_1}-(M_{11}-M_{10}M_{00}^{-1}M_{01})^{g}(M_{11}-M_{10}M_{00}^{-1}M_{01})\}\boldsymbol{z}, \tag{2.3b}$$

and hence

$$\mathcal{E}[\hat{\boldsymbol{\Theta}}_0] = \boldsymbol{\Theta}_0 + M_{00}^{-1}M_{01}\{I_{n_1}-(M_{11}-M_{10}M_{00}^{-1}M_{01})^{g}(M_{11}-M_{10}M_{00}^{-1}M_{01})\}(\boldsymbol{\Theta}_1-\boldsymbol{z}),$$

$$\mathcal{E}[\hat{\boldsymbol{\Theta}}_1] = (M_{11}-M_{10}M_{00}^{-1}M_{01})^{g}(M_{11}-M_{10}M_{00}^{-1}M_{01})\boldsymbol{\Theta}_1$$
$$+ \{I_{n_1}-(M_{11}-M_{10}M_{00}^{-1}M_{01})^{g}(M_{11}-M_{10}M_{00}^{-1}M_{01})\}\boldsymbol{z}.$$

Therefore under $\det(M_{00})\neq 0$ and ($\det(M_{11}-M_{10}M_{00}^{-1}M_{01})=0$, a necessary and sufficient condition for $\boldsymbol{\Theta}_0$ to be estimable and not to be confounded with $\boldsymbol{\Theta}_1$ is that

$$M_{00}^{-1}M_{01}\{I_{n_1}-(M_{11}-M_{10}M_{00}^{-1}M_{01})^{g}(M_{11}-M_{10}M_{00}^{-1}M_{01})\} = 0_{n_0\times n_1},$$

and hence

$$M_{01}\{I_{n_1}-(M_{11}-M_{10}M_{00}^{-1}M_{01})^{g}(M_{11}-M_{10}M_{00}^{-1}M_{01})\} = 0_{n_0\times n_1}, \tag{2.4}$$

where $0_{p\times q}$ denotes the p×q matrix with all zero. Note that under (2.4), we have

$$\mathrm{Var}[\hat{\boldsymbol{\Theta}}_0] = \sigma^2\{M_{00}^{-1} + M_{00}^{-1}M_{01}(M_{11}-M_{10}M_{00}^{-1}M_{01})^{g}M_{10}M_{00}^{-1}\}.$$

The following lemmas can easily be proved.

Lemma 2.1. *Let* $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ (= A, *say*) *be a positive semi-definite matrix with* $ac=b^2$. *Then we have*

$$A^g = \begin{pmatrix} 1/a & 0 \\ 0 & 0 \end{pmatrix} \quad \textit{if } a \neq 0,$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1/c \end{pmatrix} \quad \textit{if } c \neq 0,$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \textit{if } a = c = 0.$$

Lemma 2.2. *Let* $\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$ *(= A, say) be a positive semi-definite matrix with* $a>0$ *and* $\det(A)$ $=adf+2bce-ae^2-b^2f-c^2d=0$. *Then we have*

$$A^g = \{1/(ad-b^2)\}\begin{pmatrix} d & -b & 0 \\ -b & a & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \textit{if } ad-b^2 \neq 0,$$

$$\{1/(af-c^2)\}\begin{pmatrix} f & 0 & -c \\ 0 & 0 & 0 \\ -c & 0 & a \end{pmatrix} \quad \textit{if } ad-b^2 = 0,\ af-c^2 \neq 0,$$

$$\begin{pmatrix} 1/a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \textit{if } ad-b^2 = af-c^2 = 0.$$

3. TMDPB association scheme and its algebra

Let $S(a_1a_2)=\{(u_1^{a_1}u_2^{a_2}) \mid 1 \leq u_1 < u_2 \leq m\}$, where $a_1a_2=00,10,01,11$. Then $|S(a_1a_2)| = \binom{m}{a_1+a_2}$ (=$n(a_1a_2)$, say), where $|S|$ denotes the cardinality of a set S. Suppose a relation of association is defined among the sets $S(a_1a_2)$ in such a way that $(u_1^{a_1}u_2^{a_2}) \in S(a_1a_2)$ and $(v_1^{b_1}v_2^{b_2}) \in S(b_1b_2)$ are the α-th associates if

$$|\{u_1^{\delta(a_1)},u_2^{\delta(a_2)}\} \cap \{v_1^{\delta(b_1)},v_2^{\delta(b_2)}\}| = \min(\omega(a_1,a_2),\omega(b_1,b_2)) - \alpha,$$

where if $a_i=0$ (or $b_j=0$), then $u_i^{\delta(a_i)}$ vanishes (or $v_j^{\delta(b_j)}$ vanishes), and if $a_i \neq 0$ (or $b_j \neq 0$), then $u_i^{\delta(a_i)}=u_i$ (or $v_j^{\delta(b_j)}=v_j$). Especially, when $a_1a_2=00$ (or $b_1b_2=00$), $\{u_1^{\delta(a_1)},u_2^{\delta(a_2)}\}=\{\phi\}$ (or $\{v_1^{\delta(b_1)},v_2^{\delta(b_2)}\}=\{\phi\}$). Here $\min(a,b)$ and $\omega(a_1,a_2)$ denote the minimum value of integers a and b, and the number of non-zero elements in the vector (a_1,a_2), respectively. The scheme thus defined is called the triangular multidimensional partially balanced (TMDPB) association scheme (see Yamamoto, Shirakura and Kuwada [14,15]).

Let $A_\alpha^{(a_1a_2,b_1b_2)}$ and $D_\alpha^{(a_1a_2,b_1b_2)}$ be the $n(a_1a_2) \times n(b_1b_2)$ local association matrices and the $\tau(m) \times \tau(m)$ ordered association matrices of the TMDPB association scheme , respectively, where $\tau(m)=1+2m$

$+\binom{m}{2}$. Further let $A_\beta^{\#(a_1a_2,b_1b_2)}$ and $D_\beta^{\#(a_1a_2,b_1b_2)}$ be the matrices such that

$$A_\beta^{\#(a_1a_2,b_1b_2)} (= \{A_\beta^{\#(b_1b_2,a_1a_2)}\}') = \Sigma_\alpha\, z^{\beta\alpha}{}_{(a_1+a_2,b_1+b_2)} A_\alpha^{(a_1a_2,b_1b_2)} \quad \text{for } a_1+a_2 \leqq b_1+b_2,$$

$$D_\beta^{\#(a_1a_2,b_1b_2)} (= \{D_\beta^{\#(b_1b_2,a_1a_2)}\}') = \Sigma_\alpha\, z^{\beta\alpha}{}_{(a_1+a_2,b_1+b_2)} D_\alpha^{(a_1a_2,b_1b_2)} \quad \text{for } a_1+a_2 \leqq b_1+b_2,$$

where

$$z^{\beta\alpha}{}_{(a_1+a_2,b_1+b_2)} = \phi_\beta z_{\beta\alpha}{}^{(a_1+a_2,b_1+b_2)} / \{\binom{m}{a_1+a_2}\binom{a_1+a_2}{\alpha}\binom{m-a_1-a_2}{b_1+b_2-a_1-a_2+\alpha}\},$$

$$z_{\beta\alpha}{}^{(a_1+a_2,b_1+b_2)} = \Sigma_r \binom{a_1+a_2-\beta}{r}\binom{a_1+a_2-r}{a_1+a_2-\alpha}\binom{m-a_1-a_2-\beta+r}{r}\{\binom{m-a_1-a_2-\beta}{b_1+b_2-a_1-a_2}\binom{b_1+b_2-\beta}{b_1+b_2-a_1-a_2}\}^{1/2} / \{\binom{b_1+b_2-a_1-a_2+r}{r}\},$$

$$\phi_\beta = \binom{m}{\beta} - \binom{m}{\beta-1}.$$

Some properties of $A_\alpha^{(a_1a_2,b_1b_2)}, D_\alpha^{(a_1a_2,b_1b_2)}, A_\beta^{\#(a_1a_2,b_1b_2)}$ and $D_\beta^{\#(a_1a_2,b_1b_2)}$ are cited in the following:

$$\begin{aligned}
&A_0^{(a_1a_2,a_1a_2)} = I_{n(a_1a_2)},\quad A_\alpha^{(b_1b_2,a_1a_2)} = \{A_\alpha^{(a_1a_2,b_1b_2)}\}',\quad A_\beta^{(a_1a_2,c_1c_2)} A_\gamma^{(c_1c_2,b_1b_2)} = \Sigma_\alpha\, p(a_1a_2,b_1b_2,\alpha;c_1c_2,\beta,\gamma)\\
&\times A_\alpha^{(a_1a_2,b_1b_2)},\quad \Sigma_{a_1a_2} D_0^{(a_1a_2,a_1a_2)} = I_{\tau(m)},\quad D_\alpha^{(b_1b_2,a_1a_2)} = \{D_\alpha^{(a_1a_2,b_1b_2)}\}',\quad D_\beta^{(a_1a_2,c_1c_2)} D_\gamma^{(d_1d_2,b_1b_2)} = \delta_{c_1d_1}\delta_{c_2d_2}\\
&\times \Sigma_\alpha\, p(a_1a_2,b_1b_2,\alpha;c_1c_2,\beta,\gamma) D_\alpha^{(a_1a_2,b_1b_2)},\quad \Sigma_\beta A_\beta^{\#(a_1a_2,a_1a_2)} = I_{n(a_1a_2)},\quad A_\beta^{\#(a_1a_2,c_1c_2)} A_\gamma^{\#(c_1c_2,b_1b_2)} = \delta_{\beta\gamma}\\
&\times A_\beta^{\#((a_1a_2,b_1b_2)},\quad \operatorname{rank}(A_\beta^{\#(a_1a_2,b_1b_2)}) = \phi_\beta,\quad \Sigma_{a_1a_2}\Sigma_\beta D_\beta^{\#(a_1a_2,a_1a_2)} = I_{\tau(m)},\quad D_\beta^{\#(a_1a_2,c_1c_2)} D_\gamma^{\#(d_1d_2,b_1b_2)}\\
&= \delta_{\beta\gamma}\delta_{c_1d_1}\delta_{c_2d_2} D_\beta^{\#((a_1a_2,b_1b_2)},\quad \operatorname{rank}(D_\beta^{\#(a_1a_2,b_1b_2)}) = \phi_\beta,
\end{aligned} \tag{3.1}$$

where $\delta_{\beta\gamma}$ is the Kronecker's delta,

$$p(a_1a_2,b_1b_2,\alpha;c_1c_2,\beta,\gamma) = \Sigma_k^{(a,b)^*-\alpha} \binom{(a,b)^*-\alpha}{k}\binom{a_1+a_2-(a,b)^*+\alpha}{(a,c)^*-\beta-k} \times \binom{b_1+b_2-(a,b)^*+\alpha}{(b,c)^*-\gamma-k}\binom{m-a_1-a_2-b_1-b_2+(a,b)^*-\alpha}{c_1+c_2-(a,c)^*+\beta-(b,c)^*+\gamma+k}.$$

Here $(a,b)^* = \min(a_1+a_2,\ b_1+b_2)$, $(a,c)^* = \min(a_1+a_2,\ c_1+c_2)$ and $(b,c)^* = \min(b_1+b_2,\ c_1+c_2)$.

Let $\Omega = [D_\beta^{\#(a_1a_2,b_1b_2)} \mid a_1a_2, b_1b_2 = 00,10,01,11,\ 0 \leq \beta \leq \min(a_1+a_2,b_1+b_2)]$ which is the TMDPB association algebra generated by the linear closure of twenty six matrices $D_\beta^{\#(a_1a_2,b_1b_2)}$, and further let $\Omega_\beta = [D_\beta^{\#(a_1a_2,b_1b_2)} \mid \beta \leq \min(a_1+a_2,b_1+b_2)]$ for $\beta=0,1,2$. Then (3.1) shows the following (see [15]):

Proposition 3.1. (i) *The* Ω_β ($\beta=0,1,2$) *are the minimal two-sided ideals of* Ω, *and* $\Omega_\beta\Omega_\gamma = \delta_{\beta\gamma}\Omega_\beta$.
(ii) *The* Ω *is decomposed into the direct sum of three ideals* Ω_β, *i.e.*, $\Omega = \Omega_0 \oplus \Omega_1 \oplus \Omega_2$.
(iii) *Each ideal* Ω_β *has* $D_\beta^{\#(a_1a_2,b_1b_2)}$ *as its basis and it is isomorphic to the complete* 4×4, 3×3 *and* 1×1 *matrix algebras with multiplicities* ϕ_β *for* $\beta=0,1,2$, *respectively*.

4. Second-order designs derived from B-arrays

Consider a fractional 3^m factorial experiment. Let T be a fraction derived from a B-array of strength 4 and size N having m constraints, 3 symbols and index set $\{\lambda_{i_0 i_1 i_2} \mid i_0+i_1+i_2=4,\ i_k\geq 0\}$ which is written as BA(N,m,3,4; $\{\lambda_{i_0 i_1 i_2}\}$) for brevity. In all our evaluation, we code the three symbols of a factor as 0, 1 or 2 and employ the standard orthogonal contrasts used in the 3^m case; viz., -1, 0, 1 and 1, -2, 1 for the linear and the quadratic contrasts, respectively. Then the second-order model for T may be written as

$$\mathcal{E}[\boldsymbol{y}(T)] = E_T\boldsymbol{\Theta} \quad \text{and} \quad \mathrm{Var}[\boldsymbol{y}(T)] = \sigma^2 I_N,$$

where $\boldsymbol{\Theta}'=(\{\theta(\phi)\}; \{\theta(t^1)\}; \{\theta(t^2)\}; \{\theta(t_1^1 t_2^1)\})$. Here $\theta(\phi)$, $\theta(t^1)$, $\theta(t^2)$ and $\theta(t_1^1 t_2^1)$ denote the general mean, the linear and the quadratic components of the main effects of the factor t, and the linear by linear components of the two-factor interactions of the factors t_1 and t_2, respectively, where $1\leq t\leq m$ and $1\leq t_1 < t_2\leq m$. Then from Proposition 3.1, the information matrix $E_T'E_T (= M_T$, say) is isomorphic to $\|\kappa_\beta^{ij}\|$ (= K_β, say) for β=0,1,2, where

$$\begin{aligned}
&\kappa_0^{00} = \gamma_{400},\ \kappa_0^{01} = m^{1/2}\gamma_{310},\ \kappa_0^{02} = m^{1/2}\gamma_{301},\ \kappa_0^{03} = \{m(m-1)/2\}^{1/2}\gamma_{220},\ \kappa_0^{11} = (2\gamma_{400}+\gamma_{301})/3\\
&+(m-1)\gamma_{220},\ \kappa_0^{12} = \gamma_{310}+(m-1)\gamma_{211},\ \kappa_0^{13} = \{(m-1)/2\}^{1/2}\{2(2\gamma_{310}+\gamma_{211})/3+(m-2)\gamma_{130}\},\ \kappa_0^{22} = 2\gamma_{400}\\
&-\gamma_{301}+(m-1)\gamma_{202},\ \kappa_0^{23} = \{(m-1)/2\}^{1/2}\{2\gamma_{220}+(m-2)\gamma_{121}\},\ \kappa_0^{33} = (4\gamma_{400}+4\gamma_{301}+\gamma_{202})/9+2(m-2)(2\gamma_{220}\\
&+\gamma_{121})/3+\{(m-2)(m-3)/2\}\gamma_{040},\ \kappa_1^{00} = (2\gamma_{400}+\gamma_{301})/3-\gamma_{220},\ \kappa_1^{01} = \gamma_{310}-\gamma_{211},\ \kappa_1^{02} = (m-2)^{1/2}\{(2\gamma_{310}\\
&+\gamma_{211})/3-\gamma_{130}\},\ \kappa_1^{11} = 2\gamma_{400}-\gamma_{301}-\gamma_{202},\ \kappa_1^{12} = (m-2)^{1/2}(\gamma_{220}-\gamma_{121}),\ \kappa_1^{22} = (4\gamma_{400}+4\gamma_{301}+\gamma_{202})/9\\
&+(m-4)(2\gamma_{220}+\gamma_{121})/3-(m-3)\gamma_{040},\ \kappa_2^{00} = (4\gamma_{400}+4\gamma_{301}+\gamma_{202})/9-2(2\gamma_{220}+\gamma_{121})/3+\gamma_{040}.
\end{aligned} \tag{4.1}$$

Here $\kappa_\beta^{ij} = \kappa_\beta^{ji}$, and

$$\begin{aligned}
\gamma_{400} &= \lambda_{400}+\lambda_{040}+\lambda_{004}+4(\lambda_{310}+\lambda_{301}+\lambda_{130}+\lambda_{031}+\lambda_{103}+\lambda_{013})+6(\lambda_{220}+\lambda_{202}+\lambda_{022})+12(\lambda_{211}+\lambda_{121}+\lambda_{112}),\\
\gamma_{040} &= \lambda_{400}+\lambda_{004}-4(\lambda_{301}+\lambda_{103})+6\lambda_{202},\\
\gamma_{310} &= -\lambda_{400}+\lambda_{004}-3\lambda_{310}-2\lambda_{301}-\lambda_{130}+\lambda_{031}+2\lambda_{103}+3\lambda_{013}-3(\lambda_{220}-\lambda_{022}+\lambda_{211}-\lambda_{112}),\\
\gamma_{301} &= \lambda_{400}-2\lambda_{040}+\lambda_{004}+\lambda_{310}+4\lambda_{301}-5\lambda_{130}-5\lambda_{031}+4\lambda_{103}+\lambda_{013}-3(\lambda_{220}-2\lambda_{202}+\lambda_{022}-\lambda_{211}+2\lambda_{121}-\lambda_{112}),\\
\gamma_{130} &= -\lambda_{400}+\lambda_{004}-\lambda_{310}+2(\lambda_{301}-\lambda_{103})+\lambda_{013}+3(\lambda_{211}-\lambda_{112}),\\
\gamma_{220} &= \lambda_{400}+\lambda_{004}+2(\lambda_{310}+\lambda_{013})+\lambda_{220}-2\lambda_{202}+\lambda_{022}-2(\lambda_{211}+\lambda_{121}+\lambda_{112}),\\
\gamma_{202} &= \lambda_{400}+4\lambda_{040}+\lambda_{004}-2\lambda_{310}+4(\lambda_{301}+\lambda_{130}+\lambda_{031}+\lambda_{103})-2\lambda_{013}-3(\lambda_{220}-2\lambda_{202}+\lambda_{022})-6(\lambda_{211}+\lambda_{121}+\lambda_{112}),
\end{aligned} \tag{4.2}$$

$$\gamma_{211} = -\lambda_{400}+\lambda_{004}-2(\lambda_{301}-\lambda_{130}+\lambda_{031}-\lambda_{103})+3(\lambda_{220}-\lambda_{022}),$$

$$\gamma_{121} = \lambda_{400}+\lambda_{004}-\lambda_{310}-\lambda_{013}-2(\lambda_{220}+\lambda_{202}+\lambda_{022})+\lambda_{211}+4\lambda_{121}+\lambda_{112}$$

(see Kuwada [8]). Thus $\det(M_T)=0$ if and only if $\det(K_\beta)=0$ for some β ($\beta=0,1,2$). Note that the first, the second, the third and the last rows and columns of 4×4 matrix K_0 correspond to $\{\theta(\phi)\}$, $\{\theta(t^1)\}$, $\{\theta(t^2)\}$ and $\{\theta(t_1^1t_2^1)\}$, respectively, the first, the second and the last rows and columns of 3×3 one K_1 correspond to $\{\theta(t^1)\}$, $\{\theta(t^2)\}$ and $\{\theta(t_1^1t_2^1)\}$, respectively, and the 1×1 one K_2 corresponds to $\{\theta(t_1^1t_2^1)\}$.

5. Non-aliasing relationship for second-order model

At the beginning, we consider the case (A), i.e., the general mean and all main effects are estimable and are not confounded with the two-factor interactions. In this case, $\boldsymbol{\Theta}_0'=(\{\theta(\phi)\};\{\theta(t^1)\};\{\theta(t^2)\})$ and $\boldsymbol{\Theta}_1'=(\{\theta(t_1^1t_2^1)\})$ in (2.1). Note that M_{00} corresponds to $\{\theta(\phi)\}$, $\{\theta(t^1)\}$ and $\{\theta(t^2)\}$, and M_{11} corresponds to $\{\theta(t_1^1t_2^1)\}$. Let $K_\beta=\|K_{\beta}(ij)\|$ for $\beta=0,1$ ($i,j=0,1$), where $K_0(00)$ and $K_1(00)$ are the first 3×3 and 2×2 submatrices of K_0 and K_1, respectively, and the remainings are the submatrices of K_β of appropriate size. Then we have the following:

Theorem 5.1. *Let T be a* BA(N,m,3,4; $\{\lambda_{i_0 i_1 i_2}\}$) *with* $\det(M_T)=0$, *then a necessary and sufficient condition for the general mean and all main effects to be estimable and not to be counfounded with the two-factor interactions is that* $\det(K_\beta(00))\neq0$ *for* $\beta=0,1$ *and that* $K_\gamma(11)=0$ *if* $\det(K_\gamma)=0$ for $\gamma=0,1$.

Proof. It follows from Proposition 3.1 that M_{00} is isomorphic to $K_\beta(00)$ for $\beta=0,1$, and hence $\det(M_{00})\neq0$ if and only if $\det(K_\beta(00))\neq0$. Under $\det(M_{00})\neq0$, $M_{11}-M_{10}M_{00}^{-1}M_{01}$ is isomorphic to $K_\beta(11)-K_\beta(10)K_\beta(00)^{-1}K_\beta(01)$ for $\beta=0,1$ and K_2, and hence $\det(M_T)=0$ if and only if $\det(K_\beta(11)-K_\beta(10)K_\beta(00)^{-1}K_\beta(01))=0$ for some β ($\beta=0,1$) or $K_2=0$. While the left hand side of (2.4) is isomorphic to $K_\beta(01)\{1-(K_\beta(11)-K_\beta(10)K_\beta(00)^{-1}K_\beta(01))^g(K_\beta(11)-K_\beta(10)K_\beta(00)^{-1}K_\beta(01))\}=K_\beta(01)$ if $\det(K_\beta)=0$ and if $\det(K_\beta(00))\neq0$ ($\beta=0,1$), $\boldsymbol{O}_3$ if $\det(K_0)\neq0$, $\boldsymbol{O}_2$ if $\det(K_1)\neq0$ and vanish if $\det(K_2)\neq0$, where $\boldsymbol{O}_p=\boldsymbol{O}_{p\times1}$. Therefore (2.4) implies that $K_\gamma(11)=0$ if $\det(K_\gamma)=0$ and if $\det(K_\gamma(00))\neq0$ for $\gamma=0,1$. This completes the proof.

Note from (4.1) and (4.2) that $K_2=0$ if and only if $\lambda_{220}=\lambda_{202}=\lambda_{022}=\lambda_{211}=\lambda_{121}=\lambda_{112}=0$.

Remark 5.1. The (2.3a,b) show that $A_\beta^{\#(11,11)}\boldsymbol{\Theta}_1$ are estimable if $\det(K_\beta)\neq0$ ($\beta=0,1,2$).

Example 5.1. (I) Let T be a BA(12,4,2,4;{0,0,0,0,1,1,1,0,0,0,0,0,0,0,0}), where the index set $\{\lambda_{i_0i_1i_2}\}=\{\lambda_{400},\lambda_{040},\lambda_{004},\lambda_{310},\lambda_{301},\lambda_{130},\lambda_{031},\lambda_{103},\lambda_{013},\lambda_{220},\lambda_{202},\lambda_{022},\lambda_{211},\lambda_{121},\lambda_{112}\}$. Then from (4.1) and (4.2),

$$K_0=\begin{pmatrix}12&-4&-12&0\\-4&6&-8&0\\-12&-8&66&0\\0&0&0&0\end{pmatrix},\quad K_1=\begin{pmatrix}6&0&-4\sqrt{2}\\0&18&0\\-4\sqrt{2}&0&8\end{pmatrix},\quad K_2=0.$$

Thus $\det(K_0)=0$, $\det(K_1)\neq0$, $K_2=0$, $\det(K_\beta(00))\neq0$ for $\beta=0,1$ and $K_0(11)=0$. Therefore $\boldsymbol{\Theta}_0'=(\theta(\phi),\theta(1^1),\theta(2^1),\theta(3^1),\theta(4^1),\theta(1^2),\theta(2^2),\theta(3^2),\theta(4^2))$ is estimable and is not confounded with $\boldsymbol{\Theta}_1'=(\theta(1^12^1),\theta(1^13^1),\theta(1^14^1),\theta(2^13^1),\theta(2^14^1),\theta(3^14^1))$. Furthermore $A_1^{\#(11,11)}\boldsymbol{\Theta}_1$ is estimable.

(II) Let T be a BA(12,4,2,4;{1,0,1,0,0,0,1,0,0,0,1,0,0,0,0}), then we get

$$K_0=\begin{pmatrix}12&2&6&0\\2&9&-5&0\\6&-5&57&0\\0&0&0&16\end{pmatrix},\quad K_1=\begin{pmatrix}9&3&0\\3&9&0\\0&0&0\end{pmatrix},\quad K_2=16,$$

and hence $\det(K_0)\neq0$, $\det(K_1)=0$, $K_2\neq0$, $\det(K_\beta(00))\neq0$ for $\beta=0,1$ and $K_1(11)=0$. Therefore $\boldsymbol{\Theta}_0$ is estimable and is not confounded with $\boldsymbol{\Theta}_1$, and also $A_0^{\#(11,11)}\boldsymbol{\Theta}_1$ and $A_2^{\#(11,11)}\boldsymbol{\Theta}_1$ are estimable, where $\boldsymbol{\Theta}_0$ and $\boldsymbol{\Theta}_1$ are the same vectors as in (I).

Next we consider the case (B), i.e., all main effects are estimable and are not confounded with the general mean and the two-factor interactions. Then $\boldsymbol{\Theta}_0^{*\prime}=(\{\theta(t^1)\};\ \{\theta(t^2)\})$ and $\boldsymbol{\Theta}_1^{*\prime}=(\{\theta(\phi)\};\ \{\theta(t_1^1t_2^1)\})$ in (2.1). Let $K_0^*=P'K_0P$ $(=\|K_0^*(ij)\|$, say), $K_1^*=K_1$ $(=\|K_1^*(ij)\|$, say), and $K_2^*=K_2$, where

$$P=\begin{pmatrix}0&0&1&0\\1&0&0&0\\0&1&0&0\\0&0&0&1\end{pmatrix}.$$

Here $K_\beta^*(00)$ are the first 2x2 submatrices of K_β^* corresponding to $\{\theta(t^1)\}$ and $\{\theta(t^2)\}$, and the remainings are the submatrices of K_β^* of appropriate size for $\beta=0,1$. Then the following yields:

Theorem 5.2. *Let T be a* BA(N,m,3,4; $\{\lambda_{i_0i_1i_2}\}$) *with* $\det(M_T)=0$. *Then a necessary and sufficient condition for all main effects to be estimable and not to be confounded with the general mean* *and the two-factor interactions is that* $\det(K_\beta^*(00))\neq 0$ *for* $\beta=0,1$ *and that the last column of* K_0^* *is proportional to the third one* (*i.e., the last column of* K_0 *is proportional to the first one*) *if* $\det(K_0^*)=0$, *and* $K_1^*(11)=0$ *if* $\det(K_1^*)=0$.

Proof. From Proposition 3.1, M_{00} is isomorphic to $K_\beta^*(00)$ for $\beta=0,1$, and hence $M_{11}-M_{10}M_{00}^{-1}\times M_{01}$ is isomorphic to $K_\beta^*(11)-K_\beta^*(10)K_\beta^*(00)^{-1}K_\beta^*(01)$ and K_2^*. Thus as shown in Theorem 5.1, $\det(M_{00})\neq 0$ if and only if $\det(K_\beta^*(00))\neq 0$ for $\beta=0,1$, and under $\det(M_{00})\neq 0$, $\det(M_T)=0$ if and only if $\det(K_\beta^*(11)-K_\beta^*(10)K_\beta^*(00)^{-1}K_\beta^*(01))=0$ for some β ($\beta=0,1$) or $K_2^*=0$. We consider the case $\det(K_0^*)=0$ and $\det(K_0^*(00))\neq 0$. Let $K_0^*(11)-K_0^*(10)K_0^*(00)^{-1}K_0^*(01)=\begin{pmatrix} a^* & b^* \\ b^* & c^* \end{pmatrix}$ ($= A^*$, say) which is positive semi-definite and $a^*c^*=b^{*2}$. Now we assume $a^*=0$, then from Lemma 2.1, it holds that $A^{*g}=\begin{pmatrix} 0 & 0 \\ 0 & d^* \end{pmatrix}$, where $d^*=0$ if $c^*=0$ and $d^*=1/c^*$ if $c^*\neq 0$. Thus from (2.4), $K_0^*(01)\{I_2-(K_0^*(11)-K_0^*(10)K_0^*(00)^{-1}K_0^*(01))^g(K_0^*(11)-K_0^*(10)K_0^*(00)^{-1}K_0^*(01))\}=(\boldsymbol{x}^*, (1-d^*c^*)\boldsymbol{y}^*)$, where $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ are the 2×1 vectors corresponding to the first and the last columns of $K_0^*(01)$, respectively. Hence (2.4) implies that $\boldsymbol{x}^*=\boldsymbol{0}_2$. The (1,1)-element of $K_0^*(11)$ is $\kappa_0^{00}=N\neq 0$. On the other hand, $\boldsymbol{x}^*=\boldsymbol{0}_2$ implies that the (1,1)-element of $K_0^*(11)-K_0^*(10)K_0^*(00)^{-1}K_0^*(01)$ is $a^*=\kappa_0^{00}-\boldsymbol{x}^{*\prime}K_0^*(00)^{-1}\boldsymbol{x}^*=\kappa_0^{00}$. This is contradict. Therefore $a^*\neq 0$. From Lemma 2.1, $A^{*g}=\begin{pmatrix} 1/a^* & 0 \\ 0 & 0 \end{pmatrix}$, and hence $K_0^*(01)\{I_2-(K_0^*(11)-K_0^*(10)K_0^*(00)^{-1}K_0^*(01))^g(K_0^*(11)-K_0^*(10)K_0^*(00)^{-1}K_0^*(01))\}=(\boldsymbol{0}_2, -(b^*/a^*)\boldsymbol{x}^*+\boldsymbol{y}^*)$. Hence (2.4) implies that $a^*\boldsymbol{y}^*=b^*\boldsymbol{x}^*$. From the definition of a^*, b^* and c^*, we have

$$a^* = \kappa_0^{00} - \boldsymbol{x}^{*\prime}K_0^*(00)^{-1}\boldsymbol{x}^*,$$

$$b^* = \kappa_0^{03} - \boldsymbol{x}^{*\prime}K_0^*(00)^{-1}\boldsymbol{y}^* = \kappa_0^{03} - (b^*/a^*)\boldsymbol{x}^{*\prime}K_0^*(00)^{-1}\boldsymbol{x}^*,$$

$$c^* = \kappa_0^{33} - \boldsymbol{y}^{*\prime}K_0^*(00)^{-1}\boldsymbol{y}^* = \kappa_0^{33} - (b^*/a^*)^2\boldsymbol{x}^{*\prime}K_0^*(00)^{-1}\boldsymbol{x}^*.$$

Thus since $a^*\neq 0$ and $a^*c^*=b^{*2}$, if $\det(K_0^*)=0$ and if $\det(K_0^*(00))\neq 0$, then $a^*\kappa_0^{03}=b^*\kappa_0^{00}$ and $a^*\kappa_0^{33}=b^*\kappa_0^{30}$. Therefore (2.4) implies that the last column of K_0^* is proportional to the third one. By using the argument similar to Theorem 5.1, the (2.4) implies that $K_1^*(11)=0$ if $\det(K_1^*)=0$ and if $\det(K_1^*(00))\neq 0$. The proof is complete.

Remark 5.2. It follows from (2.3a,b) that $A_0^{\#(00,00)}\boldsymbol{\theta}^*_{10}$ and $A_0^{\#(11,11)}\boldsymbol{\theta}^*_{11}$ are estimable if $\det(K_0^*)\neq 0$, and $A_\beta^{\#(11,11)}\boldsymbol{\theta}^*_{11}$ are estimable if $\det(K_\beta^*)\neq 0$ ($\beta=1,2$), where $\boldsymbol{\theta}_{10}^{*\prime}=(\{\theta(\phi)\})$ and $\boldsymbol{\theta}_{11}^{*\prime}=(\{\theta(t_1^1t_2^1)\})$.

Example 5.2. Let T be a BA(8,4,3,4;{0,0,0,0,0,1,1,0,0,0,0,0,0,0,0}). Then from (4.1) and (4.2), we get

$$K_0^* = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 50 & -20 & 0 \\ 0 & -20 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_1^* = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad K_2^* = 0.$$

Thus $\det(K_\beta^*)=0$ for $\beta=0,1,2$, $\det(K_\gamma^*(00))\neq 0$ for $\gamma=0,1$, the last column of K_0^* is proportional to the third one, and $K_1^*(11)=0$. Therefore $\boldsymbol{\Theta}_0^{*\prime}= (\theta(1^1),\theta(2^1),\theta(3^1),\theta(4^1),\theta(1^2),\theta(2^2),\theta(3^2),\theta(4^2))$ is estimable and is not confounded with $\boldsymbol{\Theta}_1^{*\prime} =(\theta(\phi),\theta(1^12^1),\theta(1^13^1),\theta(1^14^1),\theta(2^13^1),\theta(2^14^1),\theta(3^14^1))$. However since $\det(K_\beta^*)=0$ for all β, no linear combinations of the elements of $\boldsymbol{\Theta}_1^*$ are estimable. Here $\det(K_0(00))=0$, where $K_0(00)$ is the submatrix of K_0 given in Theorem 5.1. Thus T does not satisfy the conditions of Theorem 5.1.

Finally consider the case (C), i.e., the linear components of the main effects are estimable and are not confounded with the general mean, the quadratic ones of the main effects and the two-factor interactions. Thus we have $\boldsymbol{\Theta}_0^{**\prime} = (\{\theta(t^1)\})$ and $\boldsymbol{\Theta}_1^{**\prime} = (\{\theta(\phi)\};\{\theta(t^2)\};\{\theta(t_1^1t_2^1)\})$ in (2.1). Let $K_0^{**} =Q'K_0Q$ ($= \| K_0^{**}(ij) \|$, say), $K_1^{**}=K_1$($= \| K_1^{**}(ij) \|$, say), and $K_2^{**}= K_2$, where

$$Q= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Here $K_\beta^{**}(00)$ are the first 1x1 submatrices of K_β^{**} which correspond to $\{\theta(t^1)\}$, and the remaining $K_\beta^{**}(ij)$ are the submatrices of K_β^{**} of appropriate size (β=0,1). Further let $K_\beta^{**}(-(i,j))$ be the (i,j)-cofactors of K_β^{**} for β=0,1 (i,j=0,1,2,3 if β=0; i,j=0,1,2 if β=1), where κ_β^{**ij} are the (i,j)-elements of K_β^{**}. Then we get the following:

Theorem 5.3. *Let T be a* BA(N,m,3,4; $\{\lambda_{i_0i_1i_2}\}$) *with* $\det(M_T)=0$. *Then a necessary and sufficient condition for the linear components of the main effects to be estimable and not to be confounded with the general mean, the quadratic ones of the main effects and the two-factor interactions is that* $\det(K_\beta^{**}(00))\neq 0$ *for* $\beta=0,1$ *and that* $K_0^{**}(-(3,0))=0$ *if* $\det(K_0^{**})=0$ *and if* $K_0^{**}(-(3,3))\neq 0$, $K_0^{**}(-(2,0))=0$ *if* $\det(K_0^{**})=K_0^{**}(-(3,3))=0$ *and if* $K_0^{**}(-(2,2))\neq 0$, *the third and the last columns of* K_0^{**} *are proportional to the second one (i.e., the third and the last columns of* K_0 *are proportional to the first one) if* $\det(K_0^{**})=K_0^{**}(-(3,3)) =K_0^{**}(-(2,2))=0$, *the last column of* K_1^{**} *is propor-*

tional to the second one if $\det(K_1^{**})=0$ *and if* $K_1^{**}(-(2,2))\neq 0$, $\kappa_1^{**11}=0$ *if* $\det(K_1^{**})=K_1^{**}(-(2,2))=0$ *and if* $K_1^{**}(-(1,1))\neq 0$, *and* $\kappa_1^{**11}=\kappa_1^{**22}=0$ *if* $\det(K_1^{**})=K_1^{**}(-(2,2))=K_1^{**}(-(1,1))=0$.

Proof. As shown in Theorems 5.1 and 5.2, M_{00} is isomorphic to $K_\beta^{**}(00)$ and $M_{11}-M_{10}M_{00}^{-1}M_{01}$ is isomorphic to $K_\beta^{**}(11)-K_\beta^{**}(10)K_\beta^{*}(00)^{-1}K_\beta^{**}(01)$ for $\beta=0,1$ and K_2^{**}. We consider the case $\det(K_0^{**})=0$ and $\det(K_0^{**}(00))\neq 0$. Then $K_0^{**}(11)-K_0^{**}(10)K_0^{**}(00)^{-1}K_0^{**}(01)=\begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33}\end{pmatrix}$, where

$$h_{11}=\{\kappa_0^{**00}\kappa_0^{**11}-(\kappa_0^{**01})^2\}/\kappa_0^{**00},\quad h_{12}(=h_{21})=(\kappa_0^{**00}\kappa_0^{**12}-\kappa_0^{**10}\kappa_0^{**02})/\kappa_0^{**00},$$

$$h_{13}(=h_{31})=(\kappa_0^{**00}\kappa_0^{**13}-\kappa_0^{**10}\kappa_0^{**03})/\kappa_0^{**00},\quad h_{22}=\{\kappa_0^{**00}\kappa_0^{**22}-(\kappa_0^{**02})^2\}/\kappa_0^{**00},$$

$$h_{23}(=h_{32})=(\kappa_0^{**00}\kappa_0^{**23}-\kappa_0^{**20}\kappa_0^{**03})/\kappa_0^{**00},\quad h_{33}=\{\kappa_0^{**00}\kappa_0^{**33}-(\kappa_0^{**03})^2\}/\kappa_0^{**00}.$$

Assume $h_{11}=0$. Then since $K_0^{**}(11)-K_0^{**}(10)K_0^{**}(00)^{-1}K_0^{**}(01)$ is positive semi-definite, $(K_0^{**}(11)-K_0^{**}(10)K_0^{**}(00)^{-1}K_0^{**}(01))^g=\mathrm{diag}[0, B^g]$, where $K_0^{**}(11)-K_0^{**}(10)K_0^{**}(00)^{-1}K_0^{**}(01)=\mathrm{diag}[0, B]$ and B is some 2×2 matrix. The (2.4) implies that

$$(\kappa_0^{**01}, \kappa_0^{**02}, \kappa_0^{**03})\{I_3-(K_0^{**}(11)-K_0^{**}(10)K_0^{**}(00)^{-1}K_0^{**}(01))^g(K_0^{**}(11)-K_0^{**}(10)K_0^{**}(00)^{-1}K_0^{**}(01))\}$$
$$=(\kappa_0^{**01}, (\kappa_0^{**02}, \kappa_0^{**03})\{I_2-B^gB\})$$
$$=(0, 0, 0),$$

and hence we get $\kappa_0^{**01}=0$. Thus $h_{11}=0$ implies $\kappa_0^{**11}=0$ since $\kappa_0^{**00}(=K_0^{**}(00))\neq 0$. This is contradict because $\kappa_0^{**11}=\kappa_0^{00}=N$. Therefore $h_{11}\neq 0$. After some calculations, we have

$$h_{11}h_{22}-h_{12}h_{21}=K_0^{**}(-(3,3))/\kappa_0^{**00},\quad h_{12}h_{13}-h_{11}h_{23}=K_0^{**}(-(3,2))/\kappa_0^{**00},$$

$$h_{12}h_{23}-h_{13}h_{22}=K_0^{**}(-(3,1))/\kappa_0^{**00},\quad h_{11}h_{33}-h_{13}h_{31}=K_0^{**}(-(2,2))/\kappa_0^{**00},$$

$$h_{13}h_{23}-h_{12}h_{33}=K_0^{**}(-(2,1))/\kappa_0^{**00}.$$

If $K_0^{**}(-(3,3))\neq 0$, i.e., $h_{11}h_{22}-h_{12}h_{21}\neq 0$, then from Lemma 2.2, $(K_0^{**}(11)-K_0^{**}(10)K_0^{**}(00)^{-1}K_0^{**}(01))^g=\{1/(h_{11}h_{22}-h_{12}h_{21})\}\begin{pmatrix} h_{22} & -h_{21} & 0 \\ -h_{12} & h_{11} & 0 \\ 0 & 0 & 0\end{pmatrix}$. Thus (2.4) implies that

$$(\kappa_0^{**01}, \kappa_0^{**02}, \kappa_0^{**03})\{I_3-(K_0^{**}(11)-K_0^{**}(10)K_0^{**}(00)^{-1}K_0^{**}(01))^g(K_0^{**}(11)-K_0^{**}(10)K_0^{**}(00)^{-1}K_0^{**}(01))\}$$
$$= (0, 0, \kappa_0^{**03}+\{\kappa_0^{**01}K_0^{**}(-(3,1))+\kappa_0^{**02}K_0^{**}(-(3,2))\}/K_0^{**}(-(3,3)))$$
$$=(0, 0, 0),$$

and hence we get $\kappa_0^{**01}K_0^{**}(-(3,1))+\kappa_0^{**02}K_0^{**}(-(3,2))+\kappa_0^{**03}K_0^{**}(-(3,3))=0$. While $\kappa_0^{**00}K_0^{**}(-(3,0))+\kappa_0^{**01}K_0^{**}(-(3,1))+\kappa_0^{**02}K_0^{**}(-(3,2))+\kappa_0^{**03}K_0^{**}(-(3,3))=0$. Therefore we have $K_0^{**}(-(3,0))=0$ since

$\kappa_0^{**00} \neq 0$. Similarly if $K_0^{**}(-(3,3))=0$ and if $K_0^{**}(-(2,2,))\neq 0$, then we can get $\kappa_0^{**01} K_0^{**}(-(2,1)) + \kappa_0^{**02} \times K_0^{**}(-(2,2,)) + \kappa_0^{**03} K_0^{**}(-(2,3))=0$, and hence $K_0^{**}(-(2,0))=0$. If $K_0^{**}(-(3,3))=K_0^{**}(-(2,2))=0$, from Lemma 2.2, we have $(K_0^{**}(11)-K_0^{**}(10)K_0^{**}(00)^{-1}K_0^{**}(01))^g=\mathrm{diag}[h_{11}, 0, 0]$. Thus from (2.4),

$$(\kappa_0^{**01}, \kappa_0^{**02}, \kappa_0^{**03})\{I_3-(K_0^{**}(11)-K_0^{**}(10)K_0^{**}(00)^{-1}K_0^{**}(01))^g(K_0^{**}(11)-K_0^{**}(10)K_0^{**}(00)^{-1}K_0^{**}(01))\}$$
$$= (0, \kappa_0^{**02}-\kappa_0^{**01}h_{12}/h_{11}, \kappa_0^{**03}-\kappa_0^{**01}h_{13}/h_{11})$$
$$=(0, 0, 0).$$

After some calculations, $\kappa_0^{**02}-\kappa_0^{**01}h_{12}/h_{11}=0$ and $\kappa_0^{**03}-\kappa_0^{**01}h_{13}/h_{11}=0$ mean that $\kappa_0^{**02}\kappa_0^{**11} = \kappa_0^{**01}\kappa_0^{**12}$ and $\kappa_0^{**03}\kappa_0^{**11}=\kappa_0^{**01}\kappa_0^{**13}$, respectively. While from $K_0^{**}(-(3,3))=K_0^{**}(-(2,2))=0$ and $\det(K_0^{**}(11)-K_0^{**}(10)K_0^{**}(00)^{-1}K_0^{**}(01))=0$, we get $\kappa_0^{**11}\kappa_0^{**22}=(\kappa_0^{**12})^2$, $\kappa_0^{**11}\kappa_0^{**33}=(\kappa_0^{**13})^2$ and $\kappa_0^{**11}\kappa_0^{**23}=\kappa_0^{**12}\kappa_0^{**13}$, respectively. Therefore if $K_0^{**}(-(3,3))=K_0^{**}(-(2,2))=0$, then the third and the last columns of K_0^{**} are proportional to the second. Next we consider the case $\det(K_1^{**})=0$ and $\det(K_1^{**}(00))\neq 0$. By using the argument similar to the case $\det(K_0^{*})=0$ and $\det(K_0^{*}(00))\neq 0$ in Theorem 5.2, if $K_1^{**}(-(2,2))\neq 0$, i.e., $\kappa_1^{**00}\kappa_1^{**11}-(\kappa_1^{**01})^2\neq 0$, then (2.4) implies that the last column of K_1^{**} is proportional to the second one. If $K_1^{**}(-(2,2))=0$ and $K_1^{**}(-(1,1))\neq 0$, i.e., $\kappa_1^{**00}\kappa_1^{**22} -(\kappa_1^{**02})^2 \neq 0$, it follows from Lemma 2.1 that $(K_1^{**}(11)-K_1^{**}(10)K_1^{**}(00)^{-1}K_1^{**}(01))^g = \mathrm{diag}[0, 1/\{\kappa_1^{**00}\kappa_1^{**22} -(\kappa_1^{**02})^2\}]$. Thus we get $K_1^{**}(01)\{I_2 -(K_1^{**}(11)-K_1^{**}(10)K_1^{**}(00)^{-1}K_1^{**}(01))^g (K_1^{**}(11) -K_1^{**}(10)K_1^{**}(00)^{-1}K_1^{**}(01))\}= (\kappa_1^{**01}, 0)$. The (2.4) implies that $\kappa_1^{**01}=0$ and hence $\kappa_1^{**11}(=\kappa_1^{**12}) =0$. Therefore $\kappa_1^{**11}=0$ if $K_1^{**}(-(2,2))=0$ and if $K_1^{**}(-(1,1))\neq 0$. Lastly consider the case $K_1^{**}(-(2,2)) =K_1^{**}(-(1,1))=0$. Then we have $K_1^{**}(11) -K_1^{**}(10)K_1^{**}(00)^{-1}K_1^{**}(01)=O_{2\times 2}$, and hence $K_1^{**}(01)\{I_2 -(K_1^{**}(11) -K_1^{**}(10)K_1^{**}(00)^{-1}K_1^{**}(01))^g (K_1^{**}(11)-K_1^{**}(10)K_1^{**}(00)^{-1}K_1^{**}(01))\}= K_1^{**}(01)$. The (2.4) implies that $K_1^{**}(01)=\boldsymbol{0}_2'$. From $K_1^{**}(-(2,2))=K_1^{**}(-(1,1))=0$ and $K_1^{**}(01)=\boldsymbol{0}_2'$, we get $\kappa_1^{**11}=\kappa_1^{**22} =0$. The theorem is thus established.

Remark 5.3. The (2.3a,b) show that $A_0^{\#(00,00)}\boldsymbol{\Theta}_{10}^{**}$, $A_0^{\#(01,01)}\boldsymbol{\Theta}_{11}^{**}$ and $A_0^{\#(11,11)}\boldsymbol{\Theta}_{12}^{**}$ are estimable if $\det(K_0^{**})\neq 0$, $A_1^{\#(01,01)}\boldsymbol{\Theta}_{11}^{**}$ and $A_1^{\#(11,11)}\boldsymbol{\Theta}_{12}^{**}$ are estimable if $\det(K_1^{**})\neq 0$, and $A_2^{\#(11,11)}\boldsymbol{\Theta}_{12}^{**}$ is estimable if $\det(K_2^{**})\neq 0$, where $\boldsymbol{\Theta}_{10}^{**\prime} = (\{\theta(\phi)\})$, $\boldsymbol{\Theta}_{11}^{**\prime} = (\{\theta(t^2)\})$ and $\boldsymbol{\Theta}_{12}^{**\prime} = (\{\theta(t_1^1 t_2^1)\})$.

Example 5.3. (I) Let T be a BA(2x+6y,4,3,4;{x,0,x,0,0,0,0,0,0,0,y,0,0,0,0}), where $x,y\geq 1$. Then we have

$$K_0^{**} = \begin{pmatrix} 8x & 0 & 0 & 0 \\ 0 & 2(x+3y) & 4(x+3y) & 2\sqrt{6}(x-y) \\ 0 & 4(x+3y) & 8(x+3y) & 4\sqrt{6}(x-y) \\ 0 & 2\sqrt{6}(x-y) & 4\sqrt{6}(x-y) & 4(3x+y) \end{pmatrix}, \quad K_1^{**} = \begin{pmatrix} 8y & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad K_2^{**} = 16y.$$

Thus $\det(K_\beta^{**})=0$ and $\det(K_\beta^{**}(00))\neq 0$ for $\beta=0,1$, and $K_2^{**}\neq 0$. After some calculations, we get $K_0^{**}(-(3,3))=0$, $K_0^{**}(-(2,2))\neq 0$, $K_0^{**}(-(2,0))=0$, $K_1^{**}(-(2,2))=K_1^{**}(-(1,1))=0$ and $\kappa_1^{**11}=\kappa_1^{**22}=0$. Therefore $\boldsymbol{\Theta}_0^{**\prime} = (\theta(1^1),\theta(2^1),\theta(3^1),\theta(4^1))$ is estimable and is not confounded with $\boldsymbol{\Theta}_1^{**\prime} = (\theta(\phi),\theta(1^2),\theta(2^2),\theta(3^2),\theta(4^2),\theta(1^1 2^1),\theta(1^1 3^1),\theta(1^1 4^1),\theta(2^1 3^1),\theta(2^1 4^1),\theta(3^1 4^1))$. Furthermore since $K_2^{**}\neq 0$, $A_2^{\#(11,11)}\boldsymbol{\Theta}_{12}^{**}$ is estimable, where $\boldsymbol{\Theta}_{12}^{**\prime} = (\theta(1^1 2^1),\theta(1^1 3^1),\theta(1^1 4^1),\theta(2^1 3^1),\theta(2^1 4^1),\theta(3^1 4^1))$. While since $\det(K_1^{*}(00))=0$, T does not satisfy the conditions of Theorem 5.2, where $K_1^{*}(00)$ is the submatrix of K_1^{*} in Theorem 5.2.

(II) Let T be a BA(x+8y,4,3,4;{0,x,0,0,y,0,0,y,0,0,0,0,0,0,0}), where $x\geq 0$ and $y\geq 1$. Then

$$K_0^{**} = \begin{pmatrix} 8y & 0 & 0 & 0 \\ 0 & x+8y & -4(x-4y) & 0 \\ 0 & -4(x-4y) & 16(x+2y) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_1^{**} = \begin{pmatrix} 8y & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 16y \end{pmatrix}, \quad K_2^{**} = 0.$$

Thus $\det(K_\beta^{**})=0$ for $\beta=0,1,2$, and $\det(K_\gamma^{**}(00))\neq 0$ for $\gamma=0,1$. After some calculations, we have $K_1^{**}(-(2,2))=0$, $K_1^{**}(-(1,1))\neq 0$ and $\kappa_1^{**11}=0$. If $x=0$, then $K_0^{**}(-(3,3))=K_0^{**}(-(2,2))=0$, and the third and the last columns of K_0^{**} are proportional to the second one. On the other hand, if $x\geq 1$, i.e., $x\neq 0$, then $K_0^{**}(-(3,3))\neq 0$ and $K_0^{**}(-(3,0))=0$. Therefore $\boldsymbol{\Theta}_0^{**}$ is estimable and is not confounded with $\boldsymbol{\Theta}_1^{**}$. Obviously $\det(K_1^{*}(00))=0$. Thus T does not satisfy the conditions of Theorem 5.2.

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